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The Scattering of Plane Electric Waves by Spheres

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VI. *The Scattering of Plane Electric Waves by Spheres.*

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INTRODUCTORY NOTE.

THE problem which gives its title to the present paper has been handled by various writers, notably by Lord RAYLEIGH, Sir J. J. THOMSON, and Prof. LOVE. In most cases the solutions have been expressed in a Cartesian form; but it appears to me that a marked simplification is introduced by using spherical polar co-ordinates. The preliminary analysis becomes shorter, and the conclusions are easier to interpret; in fact, the analysis is nearly as simple as in the analogous problem of electrostatics, when an electric field is disturbed by the presence of a dielectric sphere.

To obtain the requisite solutions a new general solution of the electromagnetic equations in Cartesian form is given in § 1, and is then transformed to the spherical polar form; §§ 2, 3 contain a summary of certain analytical results required in the sequel.

§ 4 contains the general solution of the problem of finding the scattered waves when a plane simple harmonic wave strikes a sphere; and in § 5 the solution is applied to the case of a small sphere. These formulæ (all of § 4 and part of § 5) were originally worked out in 1899, but publication was postponed in the hope of completing the problem of the large sphere.

In § 6 the problem of a large sphere is considered by applying to the formulæ of § 4 a method of approximation devised by Prof. H. M. MACDONALD* for dealing with waves incident from a Hertzian oscillator on a conducting sphere. The formulæ of § 6 were worked out early in 1910 and were given in my University lectures at Cambridge in that year.†

At the same time I succeeded in obtaining a different treatment (given in § 7 below) which confirmed the other results, and gave an easier process for dealing with

* 'Phil. Trans. Roy. Soc.,' A, vol. 210, 1910, p. 113. Prof. MACDONALD tells me that he had worked out (at about the same time) results in reference to the problem of § 6; but these have not been published.

† An alternative solution was obtained by Prof. J. W. NICHOLSON at about the same time; his solution starts from Sir J. J. THOMSON'S formulæ. Prof. NICHOLSON'S results originally differed from those of § 6; but on revision agreement was obtained ('Proc. Lond. Math. Soc.,' vol. 9, 1910, p. 67; vol. 11, 1912, p. 277).

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points behind the sphere. The method of this section is similar in some respects to one used by Prof. MACDONALD in a later paper.*

The formulæ of §§ 6, 7 have been delayed in publication for two reasons: in the first place I wished to obtain some confirmation from direct numerical calculation. This has now been carried out by Messrs. PROUDMAN, DOODSON and KENNEDY (of Liverpool University).† It appears that the agreement with the formulæ of § 6 is quite close (for $\kappa a = 9, 10$) from $\theta = 0°$ to $90°$, and for the Z-component up to about $120°$. The formulæ of § 7 also give good results in a cone of about $10°$ behind the sphere (that is, from $\theta = 170°$ to $180°$). It is clear, however, that an approximation suitable from $\theta = 90°$ to $170°$ (for Y) and from $\theta = 120°$ to $170°$ (for Z) has still to be obtained. But nevertheless the present approximations proved a valuable auxiliary‡ in checking and testing the numerical work.

[The paper in its original form was presented to the Society on April 13, 1916; owing to the difficulties in regard to labour and paper during the war, I was asked to condense the introductory matter of §§ 1–3. This proved to be impossible until now, on account of pressure of war-work of various kinds. In the present version § 1 has been re-written so as to reduce its bulk; in §§ 2, 3 certain formulæ have been omitted which were not used in the applications of §§ 4–6.

In re-arranging the paper it proved convenient also to number the formulæ differently. The decimal system has now been adopted; here the figure before the decimal point indicates the section of the paper in which the formula occurs. The figures following the decimal point are to be regarded as following the same order as ordinary decimal fractions. Thus (5·21) and (5·22) fall between (5·2) and (5·3), and all these formulæ occur in § 5.—*Added March* 18, 1919.]

§ 1. A GENERAL SOLUTION OF THE FUNDAMENTAL ELECTROMAGNETIC EQUATIONS.§

The fundamental equations of electromagnetic waves may be written

$$\text{(E)} \qquad \frac{K}{c^2}\frac{\partial X}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \qquad \frac{K}{c^2}\frac{\partial Y}{\partial t} = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}, \qquad \frac{K}{c^2}\frac{\partial z}{\partial t} = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}.$$

$$\text{(M)} \qquad -\mu\frac{\partial \alpha}{\partial t} = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \qquad -\mu\frac{\partial \beta}{\partial t} = \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}, \qquad -\mu\frac{\partial \gamma}{\partial t} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}.$$

* 'Phil. Trans. Roy. Soc.,' A, vol. 212, 1912, p. 299. The two methods are not identical; but they appear to yield equivalent results in all the cases to which they have been applied.

† 'Phil. Trans. Roy. Soc.,' A, vol. 217, 1917, p. 279. The calculation was originally undertaken by Dr. PROUDMAN in consequence of a suggestion made in my lectures of 1912; the work, however, proved to be longer than had been anticipated and was completed by Messrs. DOODSON and KENNEDY.

‡ See the paper last quoted, p. 292 *et seq.*

§ Revised March 18, 1919; see note at the end of the introductory remarks above.

Here (X, Y, Z) denotes the electric force, (α, β, γ) the magnetic force, K is the dielectric constant, μ is the magnetic permeability, and the axes of reference are a Cartesian right-handed system. The units adopted are those of the electromagnetic system, and c is the fundamental constant generally identified with the velocity of radiation in free space; the equations (E) are those derived from AMPÈRE's law, and the equations (M) are similarly derived from FARADAY's law, the two together constituting the circuital relations of the electromagnetic field.

It has proved possible to obtain a solution of a very general type, by assuming that

$$(1\cdot1) \qquad X = \frac{\partial P}{\partial x} - xQ, \qquad Y = \frac{\partial P}{\partial y} - yQ, \qquad Z = \frac{\partial P}{\partial z} - zQ;$$

then equations (M) yield

$$(1\cdot2) \qquad -\mu\frac{\partial \alpha}{\partial t} = y\frac{\partial Q}{\partial z} - z\frac{\partial Q}{\partial y}, \qquad -\mu\frac{\partial \beta}{\partial t} = z\frac{\partial Q}{\partial x} - x\frac{\partial Q}{\partial z}, \qquad -\mu\frac{\partial \gamma}{\partial t} = x\frac{\partial Q}{\partial y} - y\frac{\partial Q}{\partial x}.$$

Substitute from equations (1·2) in the first equation (E) and we obtain

$$(1\cdot3) \qquad \frac{\mu K}{c^2}\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial x}\left(Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}\right) - x\Delta^2 Q,$$

where Δ^2 denotes LAPLACE's operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

On comparing equations (1·1) and (1·3) they will be seen to be consistent provided that

$$(1\cdot4) \qquad \frac{\mu K}{c^2}\frac{\partial^2 P}{\partial t^2} = Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}$$

and that

$$(1\cdot5) \qquad \frac{\mu K}{c^2}\frac{\partial^2 Q}{\partial t^2} = \Delta^2 Q.$$

Thus Q must satisfy the fundamental wave-equation, which is satisfied by any component of the electric or magnetic forces (X, Y, Z) or (α, β, γ).

For our purpose it is more convenient to express the above solutions in terms of spherical polar co-ordinates r, θ, ϕ; these are supposed to form a right-handed system, when taken in this order, so as to avoid changes of sign in introducing the new co-ordinates. We write here* (R_1, R_2, R_3) for the components of electric force in the directions of r, θ, ϕ respectively; and (H_1, H_2, H_3) for the components of magnetic force.

Equations (1·1) then become

$$(1\cdot11) \qquad R_1 = \frac{\partial P}{\partial r} - rQ, \qquad R_2 = \frac{1}{r}\frac{\partial P}{\partial \theta}, \qquad R_3 = \frac{1}{r \sin\theta}\frac{\partial P}{\partial \phi}.$$

* This is done to avoid confusion with the Cartesian components used in equations (E) and (M); but in the subsequent sections we shall use (X, Y, Z) and (α, β, γ) for the spherical polar components here denoted by (R_1, R_2, R_3) and (H_1, H_2, H_3) respectively.

As regards the transformation of (1·1) to (1·11), it is sufficient to note that the gradient of P has the spherical polar components

$$\frac{\partial P}{\partial r}, \quad \frac{1}{r}\frac{\partial P}{\partial \theta}, \quad \frac{1}{r \sin\theta}\frac{\partial P}{\partial \phi},$$

and that $(r, 0, 0)$ corresponds to the Cartesian vector (x, y, z).

To obtain the formulæ corresponding to (1·2) we observe that the vector on the right is equal to the vector-product of the two vectors,

$$(x,\ y,\ z) \qquad \text{and} \qquad \left(\frac{\partial Q}{\partial x},\ \frac{\partial Q}{\partial y},\ \frac{\partial Q}{\partial z}\right);$$

and that these two are represented by

$$(r,\ 0,\ 0) \qquad \text{and} \qquad \left(\frac{\partial Q}{\partial r},\ \frac{1}{r}\frac{\partial Q}{\partial \theta},\ \frac{1}{r \sin\theta}\frac{\partial Q}{\partial \phi}\right).$$

Thus the vector-product has the spherical polar components

$$\left(0, \qquad -\frac{1}{\sin\theta}\frac{\partial Q}{\partial \phi}, \qquad \frac{\partial Q}{\partial \theta}\right).$$

Consequently equations (1·2) now become

$$(1\cdot21) \qquad -\mu\frac{\partial H_1}{\partial t} = 0, \qquad -\mu\frac{\partial H_2}{\partial t} = -\frac{1}{\sin\theta}\frac{\partial Q}{\partial \phi}, \qquad -\mu\frac{\partial H_3}{\partial t} = \frac{\partial Q}{\partial \theta},$$

while (1·4) and (1·5) give

$$(1\cdot41) \qquad \frac{\mu K}{c^2}\frac{\partial^2 P}{\partial t^2} = Q + r\frac{\partial Q}{\partial r} = \frac{\partial}{\partial r}(rQ).$$

$$(1\cdot51) \qquad \frac{\mu K}{c^2}\frac{\partial^2 Q}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial Q}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial Q}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 Q}{\partial \phi^2}$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rQ) \quad + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial Q}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 Q}{\partial \phi^2}.$$

A consideration of these formulæ suggests that further simplifications can be obtained by writing

$$(1\cdot6) \qquad P = \frac{\partial U}{\partial r}, \qquad rQ = \frac{\mu K}{c^2}\frac{\partial^2 U}{\partial t^2},$$

which together satisfy equation (1·41); and then equation (1·51) leads to the equation for U:—

$$(1\cdot7) \qquad \frac{\mu K}{c^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}.$$

Substituting from (1·6) in (1·11) and (1·21) we obtain the final expression for the field in terms of U :—

$$(1\cdot 8)\quad \left\{\begin{aligned} R_1 &= \frac{\partial^2 U}{\partial r^2} - \frac{\mu K}{c^2}\frac{\partial^2 U}{\partial t^2} \\ R_2 &= \frac{1}{r}\frac{\partial^2 U}{\partial\theta\,\partial r} \\ R_3 &= \frac{1}{\rho}\frac{\partial^2 U}{\partial\phi\,\partial r} \end{aligned}\right. \quad \left|\quad \begin{aligned} cH_1 &= 0 \\ cH_2 &= +\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{K}{c}\frac{\partial U}{\partial t}\right), \\ cH_3 &= -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{K}{c}\frac{\partial U}{\partial t}\right). \end{aligned}\right. \qquad \rho = r\sin\theta,$$

In like manner we obtain another set of solutions by making an assumption similar to (1·1) for the components of magnetic force (α, β, γ). This gives the field :—

$$(1\cdot 9)\quad \left\{\begin{aligned} R_1 &= 0 \\ R_2 &= -\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{\mu}{c}\frac{\partial V}{\partial t}\right) \\ R_3 &= +\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\mu}{c}\frac{\partial V}{\partial t}\right) \end{aligned}\right. \quad \left|\quad \begin{aligned} cH_1 &= \frac{\partial^2 V}{\partial r^2} - \frac{\mu K}{c^2}\frac{\partial^2 V}{\partial t^2} \\ cH_2 &= \frac{1}{r}\frac{\partial^2 V}{\partial\theta\,\partial r}, \\ cH_3 &= \frac{1}{\rho}\frac{\partial^2 V}{\partial\phi\,\partial r}, \end{aligned}\right. \qquad \rho = r\sin\theta,$$

where V is a second solution of equation (1·7).

It can be proved* that (1·8) gives the most general field in which the radial magnetic force (H_1) is zero, while (1·9) gives the most general field in which the radial electric force (R_1) is zero. It can also be shown that the field is uniquely determined by the value of R_1 and H_1; and accordingly the most general solution can be obtained by the superposition of (1·8) and (1·9).

§ 2. Further Specialization of the Solution of § 1.

If we superpose the fields (1·8), (1·9), and now utilize (X, Y, Z), (α, β, γ) to denote the spherical polar components of the field, we have the general solution† :—

$$(2\cdot 1)\quad \left\{\begin{aligned} X &= \frac{\partial^2 U}{\partial r^2} - \frac{\mu K}{c^2}\frac{\partial^2 U}{\partial t^2} \\ Y &= \frac{1}{r}\frac{\partial^2 U}{\partial\theta\,\partial r} - \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{\mu}{c}\frac{\partial V}{\partial t}\right), \\ Z &= \frac{1}{\rho}\frac{\partial^2 U}{\partial\phi\,\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\mu}{c}\frac{\partial V}{\partial t}\right), \end{aligned}\right. \qquad \rho = r\sin\theta,$$

* See a paper in the 'Philosophical Magazine,' July, 1919 (6th ser., vol. 38), p. 143.

† Originally worked out in 1899, and first published as a question in Part II. of the 'Mathematical Tripos,' 1910.

$$(2{\cdot}2)\qquad \begin{cases} c\alpha = \dfrac{\partial^2 V}{\partial r^2} - \dfrac{\mu K}{c^2}\dfrac{\partial^2 V}{\partial t^2} \\ c\beta = \dfrac{1}{r}\dfrac{\partial^2 V}{\partial\theta\,\partial r} + \dfrac{1}{\rho}\dfrac{\partial}{\partial\phi}\left(\dfrac{K}{c}\dfrac{\partial U}{\partial t}\right), \qquad \rho = r\sin\theta, \\ c\gamma = \dfrac{1}{\rho}\dfrac{\partial^2 V}{\partial\phi\,\partial r} - \dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left(\dfrac{K}{c}\dfrac{\partial U}{\partial t}\right), \end{cases}$$

where U, V are any two solutions of the equation

$$(2{\cdot}3)\qquad \frac{\mu K}{c^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}.$$

A solution of (2·3) which is sufficiently general for the applications in view may be found by assuming that U and V can be expressed as sums of terms of the type

$$F(r, t)\times Y(\theta, \phi).$$

It is easy to see that then (2·3) leads to the equation

$$(2{\cdot}4)\qquad \frac{r^2}{F}\left(\frac{\partial^2 F}{\partial r^2} - \frac{\mu K}{c^2}\frac{\partial^2 F}{\partial t^2}\right) = -\frac{1}{Y}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\},$$

and since the two sides of equation (2·4) are functions of r, t and of θ, ϕ respectively, it is clear that each side must be a mere constant. If we write the constant in the form $n(n+1)$, it is evident that Y must be a surface-harmonic of order n.

Accordingly in problems (such as those with which we shall be concerned in the sequel) where the *whole* of angular space is considered, the value of n must be a positive integer; for (except when n is an integer) there are no surface-harmonics which are everywhere continuous and single-valued.

Thus we may reduce our solution to the form

$$(2{\cdot}5)\qquad U \text{ or } V = \Sigma F_n(r, t)\, Y_n(\theta, \phi), \qquad n = 0, 1, 2, 3, \ldots,$$

where F_n is a solution of the equation

$$(2{\cdot}6)\qquad \frac{\partial^2 F_n}{\partial r^2} - \frac{1}{c_1^2}\frac{\partial^2 F_n}{\partial t^2} - \frac{n(n+1)}{r^2}F_n = 0,$$

and

$$c_1^2 = c^2/(\mu K).$$

The general solution of equation (2·6) is well known, and it is given by*

$$\text{(2·7)} \qquad F_n(r, t) = r^{n+1}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left\{\frac{f(c_1t-r)+g(c_1t+r)}{r}\right\},$$

where the functions f and g are arbitrary.

In the special case of *divergent waves*, the function g can be omitted in (2·7); and, if the region considered includes the origin, then $g(c_1t+r) = -f(c_1t+r)$, so as to make $F_n(r, t)$ continuous at $r = 0$.

It will be convenient to notice that in consequence of equation (2·6) the radial components of force can be written in the simpler forms

$$\text{(2·8)} \qquad X \quad \text{or} \quad c\alpha = \Sigma \frac{n(n+1)}{r^2} F_n(r, t)\, Y_n(\theta, \phi).$$

It will be noticed that we can at once determine the form (2·5) for U or V when the radial forces have been expressed in the form (2·8); this agrees with the general conclusion stated at the end of § 1, that (in spherical polar co-ordinates) the remaining components of force are completely determined when the two radial components are known.

§ 3. Special Case of Simple Harmonic Waves and the Appropriate Functions.

We assume in future that the waves are simple harmonic, of wave-length $2\pi/\kappa$ in free space; we can then suppose the time to occur only in the form of a time-factor $e^{\iota\kappa ct}$, with the usual convention that finally only the real (or the imaginary) parts of the formulæ will be used.

The functions f, g occurring in equation (2·7) above are then exponentials of the types

$$e^{\iota\kappa_1(c_1t-r)} \quad \text{and} \quad e^{\iota\kappa_1(c_1t+r)},$$

where κ_1 is given by

$$\kappa_1 c_1 = \kappa c, \quad \text{or} \quad \kappa_1 = \kappa\sqrt{(\mu K)}.$$

Thus (if we now suppress the time-factor $e^{\iota\kappa ct}$) the functions given by (2·7) are of the types

$$\text{(3·1)} \qquad r^{n+1}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{e^{-\iota\kappa_1 r}}{r}, \qquad r^{n+1}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{e^{+\iota\kappa_1 r}}{r}.$$

We shall be concerned with two special types only: (i.) divergent waves; (ii.) waves which are continuous at $r = 0$. The former of these corresponds to the

* See, for instance, LAMB'S 'Hydrodynamics,' 1906, art. 295; an alternative method of solution given in § 3 of my paper in the 'Philosophical Magazine,' quoted on p. 179 above; compare also A. E. H. LOVE ('Phil. Trans. Roy. Soc.,' A, vol. 197, 1901, pp. 9, 10).

first expression in (3·1); while the latter is found by combining the two expressions so as to yield

(3·2) $$r^{n+1}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{\sin(\kappa_1 r)}{r}.$$

Using the notation explained in (3·4) and (3·5) below, the standard functions are

$E_n(\kappa_1 r)$ for divergent waves,

and

$S_n(\kappa_1 r)$ for waves within a spherical boundary.

Consequently, for waves inside a spherical boundary, (2·5) and (2·8) can now be replaced by the forms

(3·3) $$\begin{cases} U \quad \text{or} \quad V = \Sigma S_n(\kappa_1 r)\, Y_n(\theta, \phi), \\ X \quad \text{or} \quad c\alpha = \Sigma \frac{n(n+1)}{r^2} S_n(\kappa_1 r)\, Y_n(\theta, \phi), \end{cases}$$

for divergent waves the function $S_n(\kappa_1 r)$ must be replaced by $E_n(\kappa_1 r)$.

Definitions and Properties of the Two Standard Functions $S_n(z)$, $E_n(z)$.

We write for brevity

(3·4) $$S_n(z) = z^{n+1}\left(-\frac{1}{z}\frac{d}{dz}\right)^n \left(\frac{\sin z}{z}\right)$$
$$= \frac{z^{n+1}}{1\,.\,3\,.\,5\,\ldots(2n+1)}\left\{1 - \frac{z^2}{2(2n+3)} + \frac{z^4}{2\,.\,4(2n+3)(2n+5)} - \ldots\right\}.$$

In terms of the known Bessel function we can write

(3·41) $$S_n(z) = \sqrt{\left(\frac{\pi z}{2}\right)}\, J_{n+\frac{1}{2}}(z),$$

and accordingly the function $S_n(z)$ is the same as that denoted by u in one of MACDONALD's papers.*

In the notation adopted by LAMB,† and those writers who have used LAMB's solutions as the fundamental forms, we have the identity

(3·42) $$S_n(z) = z^{n+1}\psi_n(z).$$

* 'Phil. Trans. Roy. Soc.,' A, vol. 210, 1910, p. 113. See in particular p. 115.

† 'Hydrodynamics,' 1906, Art. 287.

Similarly, we write

(3·5) $$\mathrm{E}_n(z) = z^{n+1}\left(-\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{e^{-\iota z}}{z}\right)$$
$$= \mathrm{C}_n(z) - \iota \mathrm{S}_n(z),$$

where

(3·6) $$\mathrm{C}_n(z) = z^{n+1}\left(-\frac{1}{z}\frac{d}{dz}\right)^n\left(\frac{\cos z}{z}\right)$$
$$= \frac{1\,.\,3\ldots(2n-1)}{z^n}\left\{1 - \frac{z^2}{2\,(1-2n)} + \frac{z^4}{2\,.\,4\,(1-2n)\,(3-2n)} - \ldots\right\}.$$

In terms of the K_n function (the modified Bessel function used by MACDONALD) we have the relation

(3·51) $$\mathrm{E}_n(z) = \sqrt{\left(\frac{2z}{\pi}\right)}\, e^{\frac{1}{2}(n+\frac{1}{2})\pi\iota}\mathrm{K}_{n+\frac{1}{2}}(\iota z).$$

Thus

$$\mathrm{E}_n(z) = v - \iota u$$

in terms of the notation used by MACDONALD in the paper last quoted, and in LAMB's notation

(3·7) $$\mathrm{E}_n(z) = z^{n+1}\{\Psi_n(z) - \iota\psi_n(z)\} = z^{n+1}f_n(z).$$

In consequence of the equation (2·6) we see that both $\mathrm{S}_n(z)$ and $\mathrm{E}_n(z)$ are solutions of the differential equation

(3·8) $$\frac{d^2\mathrm{S}_n}{dz^2} + \left\{1 - \frac{n(n+1)}{z^2}\right\}\mathrm{S}_n = 0,$$

The functions $\mathrm{S}_n(z)$, $\mathrm{C}_n(z)$ and $|\mathrm{E}_n(z)|$ have been tabulated from $z = 1$ to 10, and for values of n ranging from 0 to 22, by Mr. DOODSON,* and these tables have formed the basis of the numerical calculations mentioned on p. 176 above.†

It will be convenient to collect here the simple relations amongst the functions S_{n-1}, S_n, S_{n+1}, which correspond to the known results for Bessel functions, or to those given by LAMB for the equivalent function $\psi_n(z)$.

Difference Relations for the Functions S_n, E_n.

From (3·4) we see that

(3·81) $$\mathrm{S}_{n+1}(z) = -z^{n+1}\frac{d}{dz}\left\{\frac{\mathrm{S}_n(z)}{z^{n+1}}\right\} = \frac{n+1}{z}\mathrm{S}_n(z) - \frac{d\mathrm{S}_n}{dz},$$

and by using (3·5) we see that the same relation holds for $\mathrm{E}_n(z)$.

Again, it will be found that

$$\left(\frac{d}{dz} + \frac{n}{z}\right)z^{n+p} = (2n+p)\,z^{n+p-1},$$

* 'British Association Report,' 1914.

† PROUDMAN, DOODSON and KENNEDY, 'Phil. Trans. Roy. Soc.,' A, vol. 217, 1917, p. 279.

and using this in equation (3·4) we deduce that

(3·82) $$S_{n-1}(z) = \left(\frac{d}{dz} + \frac{n}{z}\right) S_n(z) = \frac{n}{z} S_n(z) + \frac{dS_n}{dz}.$$

Combining (3·81) and (3·82) we have also

(3·83) $$S_{n-1}(z) + S_{n+1}(z) = \frac{2n+1}{z} S_n(z).$$

The relations (3·82), (3·83) hold equally for $E_n(z)$ and $C_n(z)$, as may be seen from (3·5) and (3·6). As $E_n(z)$, $S_n(z)$ are independent solutions of the equation (3·8), it is evident that

$$E_n(z)\frac{dS_n}{dz} - S_n(z)\frac{dE_n}{dz} = \text{const.}$$

Now when z is small, it is easy to verify from (3·4) and (3·6) that

$$E_n(z)\frac{dS_n}{dz} = \frac{n+1}{2n+1} + O(z^2), \qquad S_n(z)\frac{dE_n}{dz} = -\frac{n}{2n+1} + O(z^2),$$

and accordingly we have

(3·84) $$E_n(z)\frac{dS_n}{dz} - S_n(z)\frac{dE_n}{dz} = 1.$$

In the discussions of § 6, when n, z are both large, it will be convenient to adopt the following notation :—

(3·85) $$|E_n(z)| = R, \quad E_n(z) = Re^{-\iota\psi}, \quad \text{so that} \quad S_n(z) = R\sin\psi, \quad C_n(z) = R\cos\psi.$$

Substituting from (3·85) in (3·84) we deduce that

(3·86) $$R^2\frac{d\psi}{dz} = 1.$$

Before leaving these preliminary formulæ it will be convenient to quote the formula for $e^{\iota\kappa z}$ in terms of our standard functions; namely

(3·9) $$e^{\iota\kappa z} = \sum_{n=0}^{\infty}(2n+1)\,\iota^n\frac{S_n(\kappa r)}{\kappa r}P_n(\mu),$$

where $z = r\cos\theta = r\mu$ and $P_n(\mu)$ is LEGENDRE's polynomial of order n.

This result follows at once from the formula given in LAMB's 'Hydrodynamics,' Art. 291, on using the relation (3·4) between $\psi_n(\kappa r)$ and $S_n(\kappa r)$, already quoted. It is of course evident that an expansion of the type (3·9) might be anticipated, since each side satisfies the wave-equation, is symmetrical about the axis of z, and is continuous at $r = 0$; the determination of the numerical coefficients may be then carried out quickly by comparing the terms in $(\kappa r\mu)^n$ on the two sides of the equation.

§ 4. PLANE ELECTROMAGNETIC WAVES INCIDENT ON A SPHERICAL OBSTACLE.

Suppose that the incident wave-train is travelling along the negative direction of the axis of z (that is, from $\theta = 0$ towards $\theta = \pi$); and that it is polarized in the plane of yz (that is, in the plane $\phi = \frac{1}{2}\pi$). Suppose further that the electric force in the wave-train has unit amplitude; then, in terms of the Cartesian specification, the incident wave is defined by*

$$X = -e^{\iota\kappa(ct+z)}, \qquad c\beta = +e^{\iota\kappa(ct+z)},$$

the remaining components of force being zero.

We must first express this wave in the standard forms of (2·1) and (2·2); we therefore introduce polar co-ordinates, and then proceed to find the radial components of force, which will suffice to determine the functions U, V.

These radial components are given by

$$(4·1) \qquad X = -\sin\theta\cos\phi\, e^{\iota\kappa r\cos\theta}, \qquad c\alpha = +\sin\theta\sin\phi\, e^{\iota\kappa r\cos\theta},$$

where the time-factor $e^{\iota\kappa ct}$ is now omitted.

Now from (3·9) we have the formula

$$e^{\iota\kappa r\cos\theta} = \sum_{n=0}^{\infty}(2n+1)\,\iota^n\frac{S_n(\kappa r)}{\kappa r}P_n(\cos\theta).$$

So, differentiating with regard to θ, we find that

$$(4·2) \qquad \sin\theta e^{\iota\kappa r\cos\theta} = -\sum_{n=1}^{\infty}(2n+1)\,\iota^{n-1}\frac{S_n(\kappa r)}{(\kappa r)^2}P'_n(\cos\theta)\sin\theta.$$

Accordingly, on substituting (4·2) in (4·1), we find that in the incident wave

$$(4·3) \qquad \begin{cases} U = -\dfrac{\sin\theta\cos\phi}{\kappa^2}\displaystyle\sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}\iota^{n-1}S_n(\kappa r)\,P'_n(\cos\theta), \\ \text{and} \\ V = +\dfrac{\sin\theta\sin\phi}{\kappa^2}\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}S_n(\kappa r)\,P'_n(\cos\theta) \end{cases}$$

by comparing the two formulæ (2·5) and (2·8).

The corresponding waves in the interior of the sphere will be given by the two functions

$$(4·4) \qquad \begin{cases} U_1 = -\dfrac{\sin\theta\cos\phi}{\kappa^2}\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}B_nS_n(\kappa_1 r)\,P'_n(\cos\theta), \\ \text{and} \\ V_1 = +\dfrac{\sin\theta\cos\phi}{\kappa^2}\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}D_nS_n(\kappa_1 r)\,P'_n(\cos\theta), \end{cases}$$

where

$$\kappa_1 = \kappa\sqrt{(\mu K)}$$

and K, μ are the fundamental constants of the spherical obstacle.

* It is assumed that in the incident wave we may take $\mu = 1$, $K = 1$, $c_1 = c$, $\kappa_1 = \kappa$.

Similarly the scattered waves will be given by the two functions

$$(4\cdot5)\quad \begin{cases} U_0 = -\dfrac{\sin\theta\cos\phi}{\kappa^2}\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}A_nE_n(\kappa r)P'_n(\cos\theta) \\ \text{and} \\ V_0 = +\dfrac{\sin\theta\sin\phi}{\kappa^2}\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\iota^{n-1}C_nE_n(\kappa r)P'_n(\cos\theta). \end{cases}$$

The boundary conditions are given by the continuity of the tangential components of electric and magnetic force at the sphere $r = a$.

It is evident from the form of equations (2·1), (2·2) that these conditions will be satisfied if we take

$$(4\cdot6)\quad \begin{cases} \dfrac{\partial U_1}{\partial r} = \dfrac{\partial U}{\partial r} + \dfrac{\partial U_0}{\partial r}, \qquad KU_1 = U + U_0 \\ \text{and} \\ \dfrac{\partial V_1}{\partial r} = \dfrac{\partial V}{\partial r} + \dfrac{\partial V_0}{\partial r}, \qquad \mu V_1 = V + V_0. \end{cases}$$

Thus we find that A_n and C_n (the coefficients in the scattered waves) are given by

$$(4\cdot7)\quad \begin{cases} \{S_n(\kappa a) + A_nE_n(\kappa a)\}\dfrac{\kappa_1}{\kappa K}\dfrac{S'_n(\kappa_1 a)}{S_n(\kappa_1 a)} = S'_n(\kappa a) + A_nE'_n(\kappa a) \\ \text{and} \\ \{S_n(\kappa a) + C_nE_n(\kappa a)\}\dfrac{\kappa_1}{\kappa\mu}\dfrac{S'_n(\kappa_1 a)}{S_n(\kappa_1 a)} = S'_n(\kappa a) + C_nE'_n(\kappa a). \end{cases}$$

The special case of a *perfectly conducting sphere* is given by making the tangential electric force zero at the sphere $r = a$; and this condition is satisfied if

$$(4\cdot61)\qquad 0 = \frac{\partial U}{\partial r} + \frac{\partial U_0}{\partial r}, \qquad 0 = V + V_0.$$

Thus we find the simpler formulæ

$$(4\cdot71)\qquad S'_n(\kappa a) + A_nE'_n(\kappa a) = 0 \qquad S_n(\kappa a) + C_nE_n(\kappa a) = 0,$$

which may be regarded as limiting forms of (4·7), when $|K| \to \infty$ and $\mu \to 0$.

The formulæ (4·5) and (4·71) lead at once to those quoted by Dr. J. PROUDMAN* in calculating the pressure of radiation due to a plane wave incident on a small conducting sphere.

In all the applications with which we shall be concerned at present the point at which the disturbance is to be calculated will be at a distance large compared with

* 'Monthly Notices of the Royal Astronomical Society,' vol. 73, 1913, p. 535.

the wave-length. Then we can simplify the general formulæ by observing that (3·5) may be replaced by the approximation

$$E_n(\kappa r) = \iota^n e^{-\iota\kappa r}$$

if $1/\kappa r$, $(1/\kappa r)^2$, &c., are neglected. Further, in the final formulæ for the forces, U_0 and V_0 occur only in the two combinations

$$M = \frac{1}{r}\frac{\partial U_0}{\partial r} = -\frac{\iota\kappa U_0}{r}, \qquad N = -\frac{1}{r}\frac{\partial V_0}{\partial r} = \frac{\iota\kappa V_0}{r},$$

where terms of the relative order $1/\kappa r$ have been rejected.

On substituting from (4·5) we find, to the same degree of accuracy,

$$(4·8)\quad \begin{cases} M = +\dfrac{1}{r}\dfrac{\partial U_0}{\partial r} = -\dfrac{\iota\kappa}{r} U_0 = \sin\theta\cos\phi\, \dfrac{e^{-\iota\kappa r}}{\kappa r} \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{2n+1}{n(n+1)} A_n P'_n(\cos\theta) \\ \text{and} \\ N = -\dfrac{1}{r}\dfrac{\partial V_0}{\partial r} = +\dfrac{\iota\kappa}{r} V_0 = \sin\theta\sin\phi\, \dfrac{e^{-\iota\kappa r}}{\kappa r} \sum\limits_{n=1}^{\infty} (-1)^n \dfrac{2n+1}{n(n+1)} C_n P'_n(\cos\theta). \end{cases}$$

Then, substituting in the general formulæ (2·1) and (2·2), we find that (to our present order of approximation) the radial components of force are zero, and that the transverse components are given by

$$(4·9)\quad \begin{cases} Y = \dfrac{\partial M}{\partial \theta} - \dfrac{1}{\sin\theta}\dfrac{\partial N}{\partial \phi} = +c\gamma \\ Z = \dfrac{1}{\sin\theta}\dfrac{\partial M}{\partial \phi} + \dfrac{\partial N}{\partial \theta} = -c\beta. \end{cases}$$

Accordingly *the electric and magnetic forces in the scattered waves are at right angles to each other and to the radius, and their magnitudes are related in the same manner as in a plane wave.*

This conclusion might very well have been anticipated; and for the case of *small* obstacles of any shape (with constants K, μ differing but little from unity) the conclusion is contained in a paper by Lord RAYLEIGH.* But I cannot find that it has been noticed for the case of spheres of any size, and of any electrical and magnetic constants.

This may serve to indicate one advantage of the formulæ in spherical polars over those in Cartesian co-ordinates.

The formulæ (4·8), (4·9), with the values of A_n, C_n given by (4·71), were those used by Messrs. PROUDMAN, DOODSON, and KENNEDY in their numerical calculations quoted in the introduction to this paper.

* 'Scientific Papers,' vol. 1, pp. 522–536. For a *small* perfectly conducting sphere the same conclusion is given by Sir J. J. THOMSON, 'Recent Researches,' p. 448.

§ 5. Spheres Small Compared with the Wave-Length.

The fundamental assumption is that $|\kappa a|$ is small enough to justify us in rejecting all but one or two terms in the power-series for $S_n(\kappa a)$ and $E_n(\kappa a)$.

It has been usual to assume further that $|\kappa_1 a|$ is correspondingly small; but Dr. Proudman has remarked that in the case of a dielectric sphere with a large value of K the second assumption need not follow from the first. It seems worth while therefore to simplify the formulæ (4·7) by expanding in powers of $|\kappa a|$, while retaining the general forms for $S_n(\kappa_1 a)$; it will be seen moreover that the resulting formulæ form a link between the results for dielectric spheres and those for conductors.

Remembering that $S_n(\kappa a)$ is of order $(\kappa a)^{n+1}$ and that $E_n(\kappa a)$ is of order $(\kappa a)^{-n}$, it is easy to see from (4·7) that in general A_n and C_n are both of order $(\kappa a)^{2n+1}$. Thus in the first approximation it will be sufficient to deal only with the coefficients A_1 and C_1; and for these we need the formulæ for $S_1(\kappa_1 a)$ and $S'_1(\kappa_1 a)$. Now from (3·4) we have

$$S_1(z) = z^2\left(-\frac{1}{z}\frac{d}{dz}\right)\left(\frac{\sin z}{z}\right) = \frac{\sin z}{z} - \cos z = \frac{\sin z}{z}(1 - z\cot z),$$

and so

$$S'_1(z) = \sin z\left(1 - \frac{1}{z^2}\right) + \frac{\cos z}{z} = \frac{\sin z}{z^2}(z^2 - 1 + z\cot z).$$

Hence

$$\frac{zS'_1(z)}{S_1(z)} = \frac{z^2 - 1 + z\cot z}{1 - z\cot z} = F(z) - 1,$$

where now

$$F(z) = z^2/(1 - z\cot z).$$

In the second place, for the functions of κa, from (3·4) and (3·6) we find the first approximations

$$S_1(z) = \tfrac{1}{3}z^2, \qquad E_1(z) = 1/z.$$

Substituting, it will be seen that for $n = 1$ the first equation in (4·7) gives

$$\left\{\frac{3A_1}{(\kappa a)^3} + 1\right\}\frac{q}{K} = 2 - \frac{3A_1}{(\kappa a)^3},$$

where

$$q = \frac{\kappa_1 a S'_1(\kappa_1 a)}{S_1(\kappa_1 a)} = F(\kappa_1 a) - 1.$$

After a little reduction the last equation gives

(5·1) $$A_1 = \tfrac{1}{3}(\kappa a)^3\,\frac{2K + 1 - F(\kappa_1 a)}{K - 1 + F(\kappa_1 a)}.$$

The second equation in (4·7) gives a similar formula for C_1, with μ taking the place of K. However, in most cases, it is sufficient to write $\mu = 1$, and then the formula simplifies further and becomes

$$C_1 = (\kappa a)^3 \left\{ \frac{1}{F(\kappa_1 a)} - \tfrac{1}{3} \right\}. \tag{5·2}$$

The two formulæ (5·1) and (5·2) are due to Dr. PROUDMAN, who has pointed out that they connect the results found by Lord RAYLEIGH for the case of dielectric spheres, and by Sir J. J. THOMSON for conducting spheres.

To deal with the case of dielectric spheres we do not regard K as large, so that $\kappa_1 a$ may be regarded as small (of the same order as κa); and then the approximations

$$1 - (\kappa_1 a) \cot(\kappa_1 a) = \tfrac{1}{3}(\kappa_1 a)^2, \qquad \text{or} \qquad F(\kappa_1 a) = 3,$$

may be used. This gives, in place of (5·1), (5·2) the simpler forms due to Lord RAYLEIGH*

$$A_1 = \tfrac{2}{3}(\kappa a)^3 \frac{K-1}{K+2}, \qquad C_1 = 0. \tag{5·21}$$

On the other hand, Sir J. J. THOMSON'S case corresponds to the assumption that K is of the form $K_1 - \iota K_2$ where K_2 is very large; then $|\kappa_1 a|$ may be regarded as large, and $\kappa_1 a$ as complex, with a negative imaginary part. Thus approximately† $\cot(\kappa_1 a) = \iota$, and so $|F(\kappa_1 a)| = |\kappa_1 a|$, which (although large) is small compared with $|K| = |\kappa_1 a|^2/(\kappa a)^2$. Hence we find from (5·1) and (5·2) the approximate results,

$$A_1 = \tfrac{2}{3}(\kappa a)^3, \qquad C_1 = -\tfrac{1}{3}(\kappa a)^3 \tag{5·22}$$

as given by Sir J. J. THOMSON.‡ Of course this pair of formulæ follow at once from (4·71), on inserting the approximations for $S_1(\kappa a)$ and $E_1(\kappa a)$ given on p. 188 above.

Dr. PROUDMAN makes the further remark that, under the conditions assumed in (5·1) and (5·2), variations in the wave-length may produce very considerable changes in the magnitudes of A_1 and C_1, on account of the presence in $F(\kappa_1 a)$ of $\cot(\kappa_1 a)$, which may vary very fast. It is of course supposed that the sphere is dielectric, otherwise $\cot(\kappa_1 a)$ could be replaced by ι, as already stated.†

It is worth while to note the simple formulæ for the scattered wave, derived from (4·9); these give, to the present order of approximation

$$\left\{\begin{aligned} Y &= +c\gamma = \frac{e^{-\iota\kappa r}}{\kappa r}\left(\tfrac{3}{2}C_1 - \tfrac{3}{2}A_1 \cos\theta\right)\cos\phi, \\ Z &= -c\beta = \frac{e^{-\iota\kappa r}}{\kappa r}\left(\tfrac{3}{2}A_1 - \tfrac{3}{2}C_1 \cos\theta\right)\sin\phi\,; \end{aligned}\right. \tag{5·3}$$

* 'Scientific Papers,' vol. 4, p. 321 (106); see also vol. 1, p. 526.

† Provided that the imaginary part of $\kappa_1 a$ exceeds π in numerical value, the error in this approximation is less than one half per cent.

‡ 'Recent Researches,' p. 448.

and so (to this order) the scattered wave is zero in the direction given by

(5·31) $$\phi = 0, \qquad \cos\theta = C_1/A_1,$$

provided that A_1 is numerically greater than C_1. Thus in Lord RAYLEIGH'S case, the direction is given by $\theta = \frac{1}{2}\pi$;* and in Sir J. J. THOMSON'S by $\theta = \frac{2}{3}\pi$.

It may be noted here that, if the sphere has a sufficiently large dielectric constant K, it may happen that A_1 is numerically less than C_1; and then the direction given by (5·31) is no longer real.

Taking K to be real (the case of a conductor having been already considered on p. 189), it is easy to see that $A_1 < C_1$ gives $F(\kappa_1 a) < 1$ (on the assumption $K > 1$). Now the function $F(z)$ steadily decreases from 3 to 0 as z varies from 0 to π; and a rough calculation shows that for $z = \frac{7}{8}\pi$, $F(z)$ is slightly less than unity. Also in order to justify the approximations used for $S_1(\kappa a)$ and $E_1(\kappa a)$, we must suppose that $(\kappa a)^2 \leqq \frac{1}{10}$.

Hence the possibility contemplated may occur if, say,

$$(\tfrac{7}{8}\pi)^2 \leqq (\kappa_1 a)^2 \leqq \pi^2, \qquad \text{and} \qquad (\kappa a)^2 \leqq \tfrac{1}{10},$$

giving

$$K \geqq 75.$$

The direction in which the scattered wave vanishes will be given by

(5·32) $$\phi = \tfrac{1}{2}\pi, \qquad \cos\theta = A_1/C_1 = 2F(\kappa_1 a)/\{3 - F(\kappa_1 a)\},$$

the final formula being simplified by remembering that K is large.

I am not aware that there is any experimental evidence showing traces of this phenomenon; in fact all the evidence shows that $\phi = 0$, $\theta = \frac{1}{2}\pi$ is not far from the truth. Thus the circumstances in actual experiments cannot have been such as to introduce the reversal of magnitude between A_1 and C_1.

(ii.) *Second Approximations.*

We proceed next to find second approximations, assuming that $|K|$ is not large; it will be necessary to retain the second terms in the series (3·4) and (3·6) for S_1 and E_1, but the first terms will suffice for S_2 and E_2.† It is easy to see that then terms of order $(\kappa a)^5$ occur in the coefficients A_1, C_1 and A_2, but that no other coefficients can contain terms of order lower than $(\kappa a)^7$.

Using now the series (3·4) for $S_1(z)$ we have

$$S_1(z) = \tfrac{1}{3}z^2(1 - \tfrac{1}{10}z^2), \qquad S'_1(z) = \tfrac{2}{3}z(1 - \tfrac{1}{5}z^2),$$

retaining the second terms only in each series. Thus we find, to the same order,

$$z\frac{S'_1(z)}{S_1(z)} = 2 - \tfrac{1}{5}z^2.$$

Also (3·6) gives similarly

$$E_1(z) = \frac{1}{z}(1 + \tfrac{1}{2}z^2), \qquad E'_1(z) = -\frac{1}{z^2}(1 - \tfrac{1}{2}z^2).$$

* A closer approximation is worked out on the next page; see formulæ (5·6) below.

† It would be possible, of course, to obtain second approximations to (5·1) and (5·2), but a glance at the formulæ shows that the work is so laborious as to be almost impracticable.

On substituting these results (4·7) becomes

$$\left(\frac{2}{K}-\frac{p^2}{5}\right)\left\{\frac{p^2}{3}\left(1-\frac{p^2}{10}\right)+\frac{A_1}{p}(1+\tfrac{1}{2}p^2)\right\} = \frac{2p^2}{3}\left(1-\frac{p^2}{5}\right)-\frac{A_1}{p}\left(1-\frac{p^2}{2}\right)$$

and

$$\left(2-\frac{Kp^2}{5}\right)\left\{\frac{p^2}{3}\left(1-\frac{p^2}{10}\right)+\frac{C_1}{p}(1+\tfrac{1}{2}p^2)\right\} = \frac{2p^2}{3}\left(1-\frac{p^2}{5}\right)-\frac{C_1}{p}\left(1-\frac{p^2}{2}\right),$$

where, for brevity, we have written

$$\kappa a = p, \qquad (\kappa_1 a)^2 = Kp^2.$$

On reducing these equations, the results are

(5·4) $$\begin{cases} A_1 = \tfrac{2}{3}p^3\left(\dfrac{K-1}{K+2}\right)\left\{1+\tfrac{3}{5}\left(\dfrac{K-2}{K+2}\right)p^2\right\} \\ \text{and} \\ C_1 = \tfrac{1}{45}(K-1)p^5. \end{cases}$$

To determine A_2, the series for $S_2(z)$ and $E_2(z)$ will be required to the first terms only; these are

$$S_2(z) = \frac{z^3}{15}, \qquad E_2(z) = \frac{3}{z^2},$$

and on substituting in (4·7), we find

(5·5) $$A_2 = \frac{p^5}{15}\left(\frac{K-1}{2K+3}\right).$$

The field of the scattered waves is then given by

(5·6) $$\begin{cases} Y = +c\gamma = \dfrac{e^{-\iota\kappa r}}{\kappa r}(\tfrac{3}{2}C_1-\tfrac{3}{2}A_1\cos\theta+\tfrac{5}{2}A_2\cos 2\theta)\cos\phi, \\ Z = -c\beta = \dfrac{e^{-\iota\kappa r}}{\kappa r}(\tfrac{3}{2}A_1-\tfrac{3}{2}C_1\cos\theta-\tfrac{5}{2}A_2\cos\theta)\sin\phi. \end{cases}$$

The field (5·6) *is accordingly zero* (*to the same degree of approximation*) *in the direction given by*

$$\phi = 0, \qquad \cos\theta = (C_1-\tfrac{5}{3}A_2)/A_1 = \tfrac{1}{15}\frac{(K-1)(K+2)}{2K+3}(\kappa a)^2.$$

This conclusion is apparently new; but it confirms an approximate result due to Lord RAYLEIGH,* according to which the scattered wave is zero in the direction given by

$$\phi = 0, \qquad \cos\theta = \tfrac{1}{25}(K-1)(\kappa a)^2,$$

when (K−1) *is treated as small.* But on the other hand, our result contradicts

* 'Scientific Papers,' vol. 1, p. 531, formula (61).

a statement made by Prof. LOVE* that there is no direction in which the scattered wave is completely cut out; however, on a closer examination of Prof. LOVE's formulæ, they appear to confirm the present conclusion.

The formulæ in question are (42) and (43) of the paper just quoted, but apparently there is a slip in (43). In the last line of (43) the factor given as 0 should really be $(z^2-y^2)/r^2$; the source of the inaccuracy being apparently in the passage from the formula (39) to (41). On introducing this additional term in the magnetic force, it appears that the electric and magnetic forces are zero in the direction given by†

$$x = 0, \qquad \frac{z}{r} = \frac{(K+2)(K-1)}{15(2K+3)}(\kappa a)^2$$

in Prof. LOVE's notation; of course "zero" means that the forces are really of order $(\kappa a)^7$ at most.

It is not difficult to prove that the formulæ (5·6) agree with those found by Lord RAYLEIGH‡ and Prof. LOVE;§ the method to be adopted is similar to that used in § 4 of my paper in the 'Philosophical Magazine' (quoted on p. 179 above). But it should be observed that in the specification of the incident wave adopted by Lord RAYLEIGH and Prof. LOVE, the electric force is parallel to the axis of y; but here the electric force is parallel to the negative direction of x. Thus if ϕ' denotes the azimuthal angle corresponding to the former specification, it is evident that $\phi' = \frac{1}{2}\pi$ corresponds to $\phi = \pi$; and accordingly we shall have in general the relation

$$\phi - \phi' = \tfrac{1}{2}\pi,$$

because both angles are measured in the right-handed sense about the axis of z.

Lord RAYLEIGH's paper contains tables and graphs from which it is easy to determine the variation of the field with θ; and in order to connect his tables with our formulæ, let us write (5·6) in the form

$$(5{\cdot}7)\quad \left\{\begin{aligned} &Y = +c\gamma = \frac{e^{-\iota\kappa r}}{\kappa r}R\cos\phi, \qquad Z = -c\beta = \frac{e^{-\iota\kappa r}}{\kappa r}S\sin\phi,\\ &\text{where}\\ &R = \tfrac{3}{2}C_1 - \tfrac{3}{2}A_1\cos\theta + \tfrac{5}{2}A_2\cos 2\theta,\\ &S = \tfrac{3}{2}A_1 - \tfrac{3}{2}C_1\cos\theta - \tfrac{5}{2}A_2\cos\theta.\end{aligned}\right.$$

Consider, first, points in the plane given by $x = 0$, in Lord RAYLEIGH's notation; this gives $\phi' = \frac{1}{2}\pi$, or $\phi = \pi$. Hence, omitting the factor $e^{-\iota\kappa r}/(\kappa r)$, the electric force is equal to R (in the direction of θ *decreasing*); and accordingly R *is represented by the graph of the Cartesian component* $(yZ-zY)/r$, given by Lord RAYLEIGH.

Secondly, consider the plane $y = 0$; that is $\phi' = 0$, or $\phi = \frac{1}{2}\pi$. Here the electric

* 'Proc. Lond. Math. Soc.,' vol. 30, 1899, p. 318.

† A numerical slip in the first line of each of the formulæ (42) and (43) has to be corrected; the correction was given in the *Errata*, vol. 31, 'Proc. Lond. Math. Soc.'

‡ 'Scientific Papers,' vol. 5, p. 559, (u) and (v).

§ Formulæ (42) and (43) of the paper just quoted (allowing for the corrections just mentioned).

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force is equal to S, perpendicular to the plane; thus S *is represented by the graph of the Cartesian component* Y.

In general, the resultant electric force is represented by

$$\frac{e^{-\iota\kappa r}}{\kappa r}\sqrt{(\mathrm{R}^2\cos^2\phi+\mathrm{S}^2\sin^2\phi)}.$$

§ 5. (iii.) *Second Approximations for Conducting Spheres.*

The foregoing algebra needs no alteration beyond replacing $\sqrt{\mathrm{K}}$ by the appropriate complex refractive index associated with the particular metal and wave-length considered. This of course assumes that $|\mathrm{K}|$ is not so large that the convergence of $\mathrm{S}_1(\kappa_1 a)$ becomes too slow to justify the approximation made above; and then the formulæ (5·4) to (5·6) provide the solution. It will be noticed that when K is complex, the equation

$$\cos\theta = \tfrac{1}{15}\frac{(\mathrm{K}-1)(\mathrm{K}+2)}{2\mathrm{K}+3}(\kappa a)^2$$

will not usually give a real value for θ: and so *there is usually no direction in which the scattered wave is zero** (or of order $(\kappa a)^7$).

It may be of interest to note here that experimental work on the scattering of light by fine particles has been carried out with silver particles suspended in water.† The corresponding values of κa seem to vary from $\frac{1}{2}$ to 2, and the value of $\sqrt{\mathrm{K}}$ is taken as $0{\cdot}2-\iota\,(3{\cdot}6)$; thus the approximations in (ii.) are not sufficient to calculate either $\mathrm{S}_n(\kappa a)$ or $\mathrm{S}_n(\kappa_1 a)$ with any accuracy.‡ In actual fact it proved necessary to use Lord Rayleigh's exact formulæ, equivalent to (4·7) above, and to go as far as $n = 4$ in the series.§

§ 6. Case of Large Perfectly Conducting Spheres.

Before proceeding to the final formulæ, it will be convenient to state certain results given by Macdonald‖ for the values of the functions $\mathrm{S}_n(z)$, $\mathrm{E}_n(z)$, when both n and z are large.

* For the case in which K − 1 is small this conclusion is given by G. W. Walker, 'Quarterly Journal of Mathematics,' vol. 30, 1899, p. 217. The formulæ given on that page agree with (5·6), when K − 1 is small; but the more general formulæ on the preceding page do not agree with (5·6) completely. I have not succeeded in tracing the discrepancy on account of the fact that G. W. Walker has omitted some of the details of his preliminary calculations.

† E. T. Paris, 'Phil. Mag.,' vol. 30 (Ser. 6), 1915, p. 459.

‡ To obtain an accuracy of 1 per cent. in $\mathrm{S}_1(z)$ by retaining two terms of the series only, it must be supposed that $|z|$ does not exceed 1·3.

§ E. T. Paris, *loc. cit.*, p. 472.

‖ 'Phil. Trans.,' vol. 210, A, 1910, p. 134: the formulæ are due to L. Lorenz originally. A very interesting method of deriving the results is given by Debye ('Math. Annalen,' Bd. 67, 1909, p. 535).

Provided that $z-(n+\frac{1}{2})$ is of an order higher than $z^{1/3}$, the formulæ are

$$(6{\cdot}1)\quad \begin{cases} & E_n(z) = Re^{-\iota\psi}, \qquad S_n(z) = R\sin\psi, \\ \text{where} & \\ & R^2 = 1/\sin\alpha, \qquad \psi = z\sin\alpha + \tfrac{1}{4}\pi - (n+\tfrac{1}{2})\alpha, \\ \text{and} & \\ & \cos\alpha = (n+\tfrac{1}{2})/z. \end{cases}$$

We shall need also the corresponding formulæ for $S'_n(z)$ and $E'_n(z)$; it will be seen that

$$\frac{E'_n(z)}{E_n(z)} = \frac{1}{R}\frac{dR}{dz} - \iota\frac{d\psi}{dz} = -\frac{1}{R^2}\left(\frac{1}{2}\frac{\cos\alpha}{\sin^2\alpha}\frac{d\alpha}{dz} + \iota\right),$$

because (3·86) gives

$$R^2\frac{d\psi}{dz} = 1.$$

Hence

$$(6{\cdot}2)\quad \begin{cases} & \dfrac{E'_n(z)}{E_n(z)} = -\dfrac{1}{R^2}(\tan\chi + \iota) = -\dfrac{\iota}{R^2}\dfrac{e^{-\iota\chi}}{\cos\chi}, \\ \text{where*} & \\ & \tan\chi = \dfrac{1}{2}\dfrac{\cos\alpha}{\sin^2\alpha}\dfrac{d\alpha}{dz} = \dfrac{1}{2}\dfrac{\cos^2\alpha}{z\sin^3\alpha}. \end{cases}$$

Similarly, we find that

$$(6{\cdot}3)\quad \frac{S'_n(z)}{S_n(z)} = \frac{1}{R}\frac{dR}{dz} + \cot\psi\frac{d\psi}{dz} = \frac{1}{R^2}(-\tan\chi + \cot\psi) = \frac{1}{R^2}\frac{\cos(\psi+\chi)}{\sin\psi\cos\chi}.$$

The formulæ to be used finally are those for A_n and C_n, given in (4·71), thus we take

$$A_n = -\frac{S'_n(z)}{E'_n(z)}, \qquad C_n = -\frac{S_n(z)}{E_n(z)},$$

where z now denotes κa. It follows from (6·1) above that

$$C_n = -\sin\psi e^{+\iota\psi} = +\tfrac{1}{2}\iota(e^{2\iota\psi}-1),$$

and using (6·2) and (6·3) we see that

$$A_n = -\iota e^{\iota(\psi+\chi)}\cos(\psi+\chi) = -\tfrac{1}{2}\iota\{e^{2\iota(\psi+\chi)}+1\}.$$

It is now an easy matter to write down an approximation to the functions M and N defined in (4·8), provided that θ is not near to 0 or π. Under these conditions we can take the approximate value

$$P_n(\cos\theta) = \sqrt{\left(\frac{2}{n\pi\sin\theta}\right)}\cos\{(n+\tfrac{1}{2})\theta - \tfrac{1}{4}\pi\},$$

* Under our conditions α is not near to zero and z is large, so that χ is small.

giving a corresponding approximation

$$P'_n(\cos\theta) = \frac{2n+1}{\sin\theta}\frac{\sin\{(n+\frac{1}{2})\theta-\frac{1}{4}\pi\}}{\sqrt{(2n\pi\sin\theta)}}.$$

Thus

$$\sin\theta\frac{2n+1}{n(n+1)}A_nP'_n(\cos\theta) = -\frac{1}{\sqrt{(2n\pi\sin\theta)}}\{e^{2\iota(\psi+\chi)}+1\}(e^{\iota\omega}-e^{-\iota\omega}),$$

where

$$\omega = (n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi,$$

and we have replaced $(n+\frac{1}{2})^2/\{n(n+1)\}$ by unity, because n is large.

Thus, on putting $\chi = 0$, we find that (4·8) gives the approximation

(6·4) $$M = -\cos\phi\frac{e^{-\iota\kappa r}}{\kappa r}\Sigma\frac{e^{\iota n\pi}}{\sqrt{(2n\pi\sin\theta)}}(e^{2\iota\psi}+1)(e^{\iota\omega}-e^{-\iota\omega}).$$

Similarly, we get the formula

(6·5) $$N = +\sin\phi\frac{e^{-\iota\kappa r}}{\kappa r}\Sigma\frac{e^{\iota n\pi}}{\sqrt{(2n\pi\sin\theta)}}(e^{2\iota\psi}-1)(e^{\iota\omega}-e^{-\iota\omega}).$$

With series of this type, the leading part is found by making the index of the exponential stationary (regarded as a function of n). Now in both M and N there is one index only, $\sigma = 2\psi+n\pi-\omega$, which can be stationary: and the condition is

$$\frac{d\sigma}{dn} = 2\frac{d\psi}{dn}+\pi-\theta = 0.$$

Now, from (6·1)

$$\frac{d\psi}{dn} = \{z\cos\alpha-(n+\tfrac{1}{2})\}\frac{d\alpha}{dn}-\alpha = -\alpha,$$

and so the leading terms in (6·4) and (6·5) arise from taking

$$2\alpha = \pi-\theta, \quad \text{or} \quad n+\tfrac{1}{2} = z\sin\tfrac{1}{2}\theta.$$

The corresponding value of the index σ is then

$$\sigma_0 = 2\psi+n\pi-\omega = 2z\sin\alpha+\tfrac{1}{2}\pi-(2n+1)\alpha+n\pi-(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi,$$
$$= 2z\cos\tfrac{1}{2}\theta+\tfrac{1}{4}\pi.$$

To determine the form of the index near to this special value of n we take

$$\frac{d^2\sigma}{dn^2} = -2\frac{d\alpha}{dn} = \frac{2}{z\sin\alpha} = \frac{2}{z\cos\frac{1}{2}\theta},$$

and then we find the approximate formulæ

(6·51) $$\sigma = \sigma_0+\frac{(n-n_0)^2}{z\cos\frac{1}{2}\theta},$$
where
$$n_0+\tfrac{1}{2} = z\sin\tfrac{1}{2}\theta.$$

The leading parts of M, N are accordingly given by the approximations

(6·6) $$M = +\cos\phi\frac{e^{-\iota\kappa r}}{\kappa r}Q, \qquad N = -\sin\phi\frac{e^{-\iota\kappa r}}{\kappa r}Q,$$

where

$$Q = \Sigma\frac{e^{\iota\sigma}}{\sqrt{(2n\pi\sin\theta)}},$$

and σ has the value given in (6·51).

The value of Q is approximately equal to the integral

(6·61) $$\int_{-\infty}^{\infty}\frac{e^{\iota\sigma}\,dx}{\sqrt{(2n_0\pi\sin\theta)}},$$

where

$$n = n_0+x, \qquad \sigma = \sigma_0 + x^2/(z\cos\tfrac{1}{2}\theta).$$

Thus, approximately

$$Q = \frac{e^{\iota\sigma_0}}{\sqrt{(2n_0\pi\sin\theta)}}\sqrt{\left(\frac{\pi z\cos\frac{1}{2}\theta}{-\iota}\right)}$$
$$= \frac{e^{\iota(\sigma_0+\frac{1}{4}\pi)}}{2\sin\frac{1}{2}\theta} = \frac{\iota e^{\iota(2z\cos\frac{1}{2}\theta)}}{2\sin\frac{1}{2}\theta}.$$

Accordingly, to the same degree of approximation, we can take

(6·62) $$\frac{dQ}{d\theta} = \tfrac{1}{2}ze^{2\iota z\cos\frac{1}{2}\theta}.$$

Then the components of force are given, as in (4·9), by

(6·63) $$\begin{cases} Y = +c\gamma = \dfrac{\partial M}{\partial\theta} - \dfrac{1}{\sin\theta}\dfrac{\partial N}{\partial\phi} = \cos\phi\dfrac{e^{-\iota\kappa r}}{\kappa r}\dfrac{dQ}{d\theta}, \\ Z = -c\beta = \dfrac{1}{\sin\theta}\dfrac{\partial M}{\partial\phi} + \dfrac{\partial N}{\partial\theta} = -\sin\phi\dfrac{e^{-\iota\kappa r}}{\kappa r}\dfrac{dQ}{d\theta}, \end{cases}$$

where differential coefficients with respect to ϕ are small compared with those with respect to θ, and so have been rejected. Thus, using (6·62), we have the approximations to the forces in the scattered waves

(6·7) $$\begin{cases} Y = +c\gamma = \cos\phi\dfrac{a}{2r}e^{2\iota\kappa a\cos\frac{1}{2}\theta-\iota\kappa r}, \\ Z = -c\beta = -\sin\phi\dfrac{a}{2r}e^{2\iota\kappa a\cos\frac{1}{2}\theta-\iota\kappa r}, \end{cases}$$

assuming that θ is neither near to 0 nor to π.

When θ is small, the approximation to $P_n(\cos\theta)$ must be taken as

$$P_n(\cos\theta) = J_0\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

and so

$$\sin\theta P'_n(\cos\theta) = (n+\tfrac{1}{2})\cos\tfrac{1}{2}\theta\,J_1\{(2n+1)\sin\tfrac{1}{2}\theta\}.$$

Thus we now write

$$\sin\theta \frac{2n+1}{n(n+1)} A_n P'_n(\cos\theta) = -\iota(e^{2\iota\psi}+1)\cos\tfrac{1}{2}\theta\, J_1\{(2n+1)\sin\tfrac{1}{2}\theta\},$$

and, proceeding as before, we are led to the conclusion that α is near to $\frac{1}{2}\pi$; thus the value of n/z is small, in the parts of the series which contribute the principal part of the sum. Then we can replace (6·1) by the approximate formulæ

$$(6\cdot8)\quad \begin{cases} \alpha = \frac{1}{2}\pi - (n+\frac{1}{2})/z \\ \text{and} \\ \psi = z + \frac{1}{4}\pi - (n+\frac{1}{2})(\frac{1}{2}\pi) + \frac{1}{2}(n+\frac{1}{2})^2/z. \end{cases}$$

Thus

$$2\psi + n\pi = 2z + (n+\tfrac{1}{2})^2/z.$$

Hence the approximation corresponding to (6·4) is now

$$(6\cdot81)\qquad M = -\iota\cos\tfrac{1}{2}\theta\cos\phi\,\frac{e^{-\iota\kappa r}}{\kappa r}\,\Sigma\, e^{2\iota z+\iota(n+\frac{1}{2})^2/z} J_1\{2n+1)\sin\tfrac{1}{2}\theta\}.$$

In like manner the value of N is found to differ from (6·81) only in having $+\sin\phi$ as a factor instead of $-\cos\phi$.

In the series (6·81) the value of n may be supposed to vary from 0 to ∞; and so we obtain the principal part of the sum by using the integral

$$(6\cdot82)\qquad e^{2\iota z}\int_0^\infty e^{\iota\zeta^2/z} J_1(2\zeta\sin\tfrac{1}{2}\theta)\,d\zeta;$$

and when $\sin\frac{1}{2}\theta$ is very small the value of (6·82) is approximately equal to

$$e^{2\iota z}(\tfrac{1}{2}\iota z\sin\tfrac{1}{2}\theta).$$

Thus

$$(6\cdot83)\quad \begin{cases} M = +\frac{1}{4}z\sin\theta\cos\phi e^{2\iota z}\frac{e^{-\iota\kappa r}}{\kappa r}, \\ \text{and} \\ N = -\frac{1}{4}z\sin\theta\sin\phi e^{2\iota z}\frac{e^{-\iota\kappa r}}{\kappa r}. \end{cases}$$

Accordingly the components of force are now found to be

$$(6\cdot9)\quad \begin{cases} Y = +c\gamma = \dfrac{\partial M}{\partial\theta} - \dfrac{1}{\sin\theta}\dfrac{\partial N}{\partial\phi} = \quad \frac{1}{2}z\cos\phi\,\dfrac{e^{2\iota z-\iota\kappa r}}{\kappa r}, \\ Z = -c\beta = \dfrac{1}{\sin\theta}\dfrac{\partial M}{\partial\phi} + \dfrac{\partial N}{\partial\theta} = -\frac{1}{2}z\sin\phi\,\dfrac{e^{2\iota z-\iota\kappa r}}{\kappa r}. \end{cases}$$

These results in (6·9) are precisely the same as would be found by writing $\theta = 0$ in the approximations (6·7), and *accordingly the formulæ* (6·7) *do remain valid right up to the axis* $\theta = 0$.

When θ is nearly equal to π, the calculation on the present lines becomes more difficult,* and we shall accordingly obtain the corresponding approximation by a different process in the next section (§ 7).

It appears that for the special value $\kappa a = 10$, the formulæ (6·7) do give the forces with a fair degree of accuracy up to an angle $\theta = \frac{2}{3}\pi$; the approximation in fact appears to be better than might have been expected. [See p. 176 above.]

§ 7. Alternative Method, applicable to any Conductor whose Dimensions are Large compared with the Wave-length.

It follows at once from Green's theorem that if u, v are solutions of the equations

$$\Delta^2 u + \kappa^2 u = 0, \qquad \Delta^2 v + \kappa^2 v = 0,$$

at points within a closed simple surface S, then

$$(7·1) \qquad \int \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS = 0,$$

where the integral is taken over the surface S ($d\nu$ being the element of outward normal), and it is supposed that u, v are both free from singularities in the interior of S.

Similarly if u has no singularities and v is a solution which behaves like $e^{-\iota\kappa R}/R$ near a particular point P (R denoting the distance measured from P), we see that

$$(7·11) \qquad \int \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS = -4\pi u_P,$$

provided that P is inside the surface S.

Equations of similar forms apply when the space considered is *outside* the surface S; but then the sign of the last equation (7·11) is reversed, giving

$$(7·12) \qquad \int \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS = +4\pi u_P;$$

it is then necessary to assume also that at infinity, u, v both correspond to divergent waves (unless it is known that u tends to zero more rapidly than $1/r$).

* Compare Macdonald, *loc. cit.*, pp. 120–122. The cause of the difficulty is to be found in the fact that now the stationary value may be expected to arise from values of n for which α is small. Then n is nearly equal to z, and in all such cases more complicated analysis is inevitable. In fact the approximations to $S_n(z)$ and $E_n(z)$ require to be modified by different formulæ corresponding to the cases $n > z$, $n < z$ and to the cases in which $|n - z|$ is of order $z^{1/3}$ or of lower order.

Let us now consider the problem of waves incident from some source (or sources) and reflected from the surface S. Let u denote any Cartesian component of force in the incident wave, and let the point P be outside S. Then (7·1) applies, and so

$$\int \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) dS = 0,$$

because u has no singularity inside S.

If u' denotes the corresponding component of force in the reflected wave, we have from (7·12)

$$\int \left(u' \frac{\partial v}{\partial \nu} - v \frac{\partial u'}{\partial \nu} \right) dS = 4\pi u'_P,$$

because u' has no singularity outside S (and u' will correspond to a divergent wave at infinity).

By addition we have the result

$$(7·13) \qquad 4\pi u'_P = \int \left\{ (u+u') \frac{\partial v}{\partial \nu} - v \frac{\partial}{\partial \nu} (u+u') \right\} dS.$$

where v is taken to be $e^{-\iota\kappa R}/R$.

Now $u+u' = w$ gives the corresponding component of force in the complete wave; and this accordingly satisfies certain known relations at the surface S (the exact form depending on the physical properties of S). It must, however, be clearly understood that we cannot usually obtain both w and $\frac{\partial w}{\partial \nu}$ by any simple methods, any more than the analogous problems of electrostatics can be solved by a mere appeal to GREEN's Theorem.

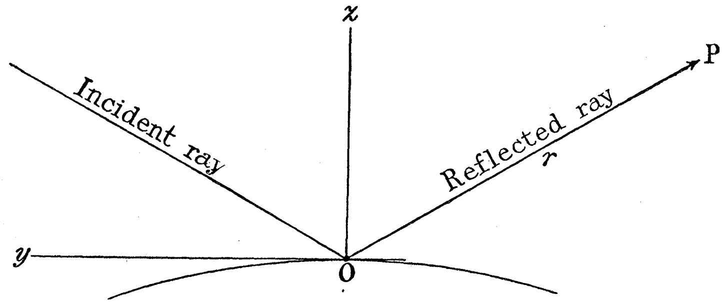
However, we can obtain *an approximate solution, suitable to the problem of short wave-lengths*, by assuming that near the reflecting surface, the character of u' can be determined from that of u by the rules of elementary geometrical optics. Thus we treat the reflected wave as derived from the incident by simple reflexion *in the tangent-plane* at the point of incidence. Making this hypothesis it is an easy matter to construct both w and $\frac{\partial w}{\partial \nu}$, when the form of u is given.

It will be noticed that we shall have $w = 0$, $\frac{\partial w}{\partial \nu} = 0$ at all points within the geometrical shadow; and so the final integral (7·13) extends only over the illuminated side of the surface S.

Suppose now that we consider electric waves incident on a simple convex conducting surface; and take an origin O on the surface such that OP is the reflected ray (in the sense of geometrical optics). Take the plane of incidence as the plane of yz, and the normal at O as the axis of z.

Then in the immediate neighbourhood of O we can represent the incident wave by the components of electric force

$$(A, \ B, \ C)\, e^{\iota\kappa(my+nz)}$$



where $m^2+n^2 = 1$, so that m is the cosine and n the sine of the angle of incidence. We have also the relation

$$Bm+Cn = 0$$

because the electric force is perpendicular to the incident ray.

The corresponding reflected wave has the components

$$(A', \ B', \ C')\, e^{\iota\kappa(my-nz)}$$

where

$$A+A' = 0, \qquad B+B' = 0,$$

and

$$B'm-C'n = 0,$$

so that

$$C'-C = 0\,;$$

these results follow by making the tangential components of force zero on the tangent-plane $z = 0$ (instead of at the surface).

It is now clear that, at the point O, the components of the total force will be

(7·2) $$(0, \ 0, \ 2C),$$

and the normal differential coefficients of the total force will be

(7·21) $$\iota\kappa n\,(A-A', \ B-B', \ C-C') = 2\iota\kappa n\,(A, \ B, \ 0).$$

We shall now insert these values for w and $\frac{\partial w}{\partial \nu}$ in the general formula (7·13), including also the factor $e^{\iota\kappa(my+nz)}$, on account of phase-differences at points near to O. Since reflexion takes place only from the immediate neighbourhood of O, the error introduced by this simplification will be small.

Since the co-ordinates of P are $(0, -mr, nr)$, the value of R is given by

$$R^2 = x^2 + (y+mr)^2 + (z-nr)^2,$$
$$= r^2 + 2r(my-nz) + x^2 + y^2 + z^2,$$

where (x, y, z) is a point on the surface near to O. Thus, when r is very large in comparison with the dimensions of the surface (as we assumed in the previous investigations, §§ 4–6), we can use the approximate formula

(7·3) $$R = r + my - nz.$$

Thus we can write in (7·13)

$$v = \frac{e^{-\iota\kappa(r+my-nz)}}{r} = v_0 e^{-\iota\kappa(my-nz)},$$

if v_0 is the value of v at O. Then the most important term in $\frac{\partial v}{\partial \nu}$ is seen to be

$$\frac{\partial v}{\partial z} = \iota\kappa n v_0 e^{-\iota\kappa(my-nz)}.$$

Accordingly the components of electric force in the reflected wave will be given by the approximation

(7·4) $$\left\{\begin{aligned} X &= -\frac{v_0}{4\pi}\int(2\iota\kappa n A)\, e^{2\iota\kappa nz}\, dS, \\ Y &= -\frac{v_0}{4\pi}\int(2\iota\kappa n B)\, e^{2\iota\kappa nz}\, dS, \\ Z &= +\frac{v_0}{4\pi}\int(2\iota\kappa n C)\, e^{2\iota\kappa nz}\, dS. \end{aligned}\right.$$

To evaluate the integrals in (7·4) we must write out the equation to the surface in the approximate form

$$2z = -(\alpha x^2 + 2\beta xy + \gamma y^2).$$

Then*

$$\int e^{-\iota\kappa n(\alpha x^2+2\beta xy+\gamma y^2)}\, dS = \frac{\pi}{\iota\kappa n}\frac{1}{\sqrt{(\alpha\gamma-\beta^2)}}.$$

Now $\alpha\gamma-\beta^2$ is the absolute (or Gaussian) curvature of the surface at the point O; and since the surface is supposed convex, we represent this curvature by $1/\rho^2$.

* This is most easily found by taking the integral as $\lim\limits_{\theta\to 0}\int e^{-(\theta+\iota\kappa)n(\alpha x^2+2\beta xy+\gamma y^2)}\, dS$.

Thus we get

$$\iota\kappa n \int e^{2\iota\kappa n z}\, dS = \pi\rho. \tag{7·41}$$

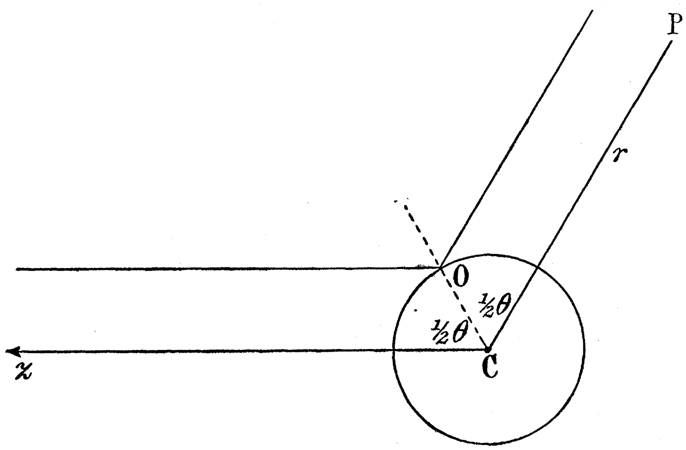
Hence, using (7·4), we see that the principal part of the reflected wave is given by

$$(X, Y, Z) = (-A, -B, +C)\frac{\rho}{2r}e^{-\iota\kappa r}. \tag{7·5}$$

In order to interpret (7·5) for any axes of co-ordinates, we need only notice that $(-A, -B, +C)$ represents a force numerically equal to the force in the incident wave; and that the new force is perpendicular to the reflected ray, arranged in such a way that the tangential components are opposite to those in the incident wave.

We can apply the formula (7·5) to the problem of § 6 at once; clearly $\rho = a$.

The point of incidence corresponding to the scattered wave (θ, ϕ) is given by $(\frac{1}{2}\theta, \phi)$.



Then the incident wave at O has the components of electric force

$$-\cos\phi e^{\iota\kappa a\cos\frac{1}{2}\theta} \text{ in the plane of incidence,}$$

and

$$+\sin\phi e^{\iota\kappa a\cos\frac{1}{2}\theta} \text{ perpendicular to the plane of incidence.}$$

Further, the r of formula (7·5) is measured from O; to compare with § 6, we take r to be the distance CP, measured from the centre of the sphere. Thus we are to replace r in (7·5) by $r - a\cos\frac{1}{2}\theta$. Accordingly the components of force at P, in the reflected wave, are

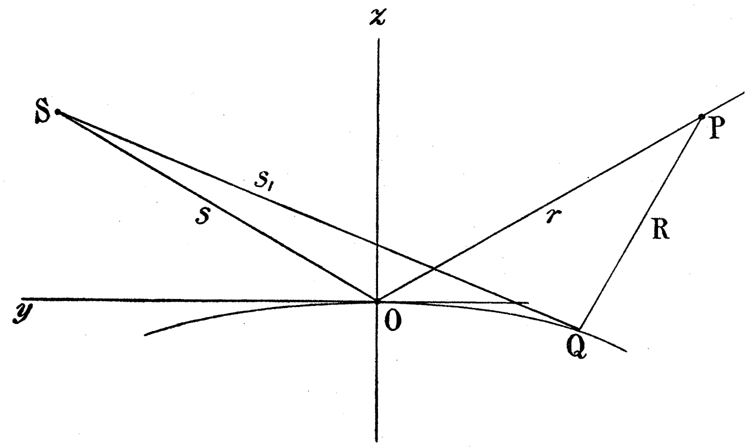
$$+\frac{a}{2r}\cos\phi e^{2\iota\kappa a\cos\frac{1}{2}\theta-\iota\kappa r} \text{ perpendicular to } r \text{ in the plane ZCP,}$$

and

$$-\frac{a}{2r}\sin\phi e^{2\iota\kappa a\cos\frac{1}{2}\theta-\iota\kappa r} \text{ perpendicular to the plane ZCP.}$$

These results agree with (6·7) and (6·9) of § 6 above.

It is easy to modify the general formulæ (7·5) so as to cover the case of waves incident from a point source (say at distance s from the point of incidence).



Then

$$s_1^2 = x^2 + (ms-y)^2 + (ns-z)^2$$
$$= s^2 - 2s(my+nz) + x^2 + y^2 + z^2.$$

Thus with the usual approximation of geometrical optics

$$s_1 = s - (my+nz) + \frac{1}{2s}\{x^2 + (ny-mz)^2\}.$$

Similarly

$$R = r + my - nz + \frac{1}{2r}\{x^2 + (ny+mz)^2\}.$$

Now z is of the second order in comparison with x, y; and so we can write

$$s_1 + R = s + r - n(\alpha x^2 + 2\beta xy + \gamma y^2) + \left(\frac{1}{2r} + \frac{1}{2s}\right)(x^2 + n^2y^2).$$

Thus here we find

$$\int e^{-\iota\kappa(s_1+R)}\, dS = e^{-\iota\kappa(s+r)}\frac{\pi}{\iota\kappa n\sigma}$$

where

$$\sigma^2 = \left\{\tfrac{1}{2}\left(\frac{1}{r}+\frac{1}{s}\right) - n\alpha\right\}\left\{\tfrac{1}{2}\left(\frac{1}{r}+\frac{1}{s}\right) - \frac{\gamma}{n}\right\} - \beta^2$$
$$= \tfrac{1}{4}\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right),$$

r_1, r_2 being the distances of the focal lines (of geometrical optics) from the point of incidence.

Thus now the principal parts of the reflected wave are given by

$$(-A, \ -B, \ +C)\, e^{-\iota\kappa r} \Big/ \left\{\left(1 - \frac{r}{r_1}\right)\left(1 - \frac{r}{r_2}\right)\right\}^{1/2};$$

assuming that r is not close either to r_1, or to r_2.

The results of the foregoing analysis depend on the tacit assumption that n is not zero; and as a consequence the character of the approximations will change when n

is small. That is, near the edge of the shadow, in the ordinary phrase of geometrical optics.

In the application to the sphere, the region excluded by this condition corresponds to values of θ nearly equal to π; and in this region the specification of the scattered wave by means of Cartesian co-ordinates seems simplest.

We consider then the incident wave as specified in Cartesian form by

$$X = -e^{\iota\kappa z}, \qquad Y = 0, \qquad Z = 0, \tag{7·6}$$

and consider the approximation to the reflected wave incident at the point (al, am, an) on the sphere for which the direction-cosines of the normal are l, m, n. The expressions will be of the form

$$(X', \ Y', \ Z') = (A', \ B', \ C')e^{\iota\kappa\zeta} \tag{7·61}$$

where (treating the tangent-plane as the reflecting surface)

$$\zeta = z+p\,(a-lx-my-nz)$$

and p is determined by

$$(-lp)^2+(-mp)^2+(1-np)^2 = 1.$$

Thus

$$p = 2n.$$

Further, the resultant of (X, Y, Z) and (X′, Y′, Z′) at the point of incidence must be along the normal; and so

$$\frac{A'-1}{l} = \frac{B'}{m} = \frac{C'}{n}.$$

Also (A′, B′, C′) is perpendicular to the reflected ray; and so

$$A'\,(-2nl)+B'\,(-2nm)+C'\,(1-2n^2) = 0.$$

Hence

$$(7·62)\begin{cases} \dfrac{A'-1}{l} = \dfrac{B'}{m} = \dfrac{C'}{n} = \dfrac{2nl}{n-2n} = -2l, \\ \text{or} \\ A' = 1-2l^2, \quad B' = -2lm, \quad C' = -2ln. \end{cases}$$

The components X+X′, Y+Y′, Z+Z′ at the point of incidence are accordingly equal to

$$-2\,(l^2, \ lm, \ ln)\,e^{\iota\kappa an}. \tag{7·63}$$

We have still to evaluate the normal differential coefficients, which are found to be

$$\iota\kappa n\,(-1-A', \ -B', \ -C')\,e^{\iota\kappa an} = 2\iota\kappa n\,(l^2-1, \ lm, \ ln)\,e^{\iota\kappa an}. \tag{7·64}$$

The value of R is now seen to be given by

$$R^2 = (x-al)^2+(y-am)^2+(z-an)^2 = r^2-2a\,(lx+my+nz)+a^2.$$

Thus, when we regard a/r as small, we may take

$$(7\cdot65)\quad \begin{cases} \mathrm{R} = r - a\,(l\sin\theta\cos\phi + m\sin\theta\sin\phi + n\cos\theta), \\ \text{and} \\ \dfrac{\partial\mathrm{R}}{\partial\nu} = -(l\sin\theta\cos\phi + m\sin\theta\sin\phi + n\cos\theta). \end{cases}$$

Now, in applying (7·13), $\dfrac{\partial\mathrm{R}}{\partial\nu}$ occurs only in the coefficients and not in the exponential index: thus we can get the first approximation by putting $\theta = \pi$ in $\dfrac{\partial\mathrm{R}}{\partial\nu}$; this gives the value

$$(7\cdot66)\qquad \frac{\partial\mathrm{R}}{\partial\nu} = n.$$

Thus, to our degree of accuracy

$$\frac{1}{4\pi}\left(w\frac{\partial v}{\partial\nu} - v\frac{\partial w}{\partial\nu}\right) = -\frac{v}{4\pi}\left(\iota\kappa w\frac{\partial\mathrm{R}}{\partial\nu} + \frac{\partial w}{\partial\nu}\right),$$

$$= -\frac{v}{4\pi}\left(\iota\kappa n w + \frac{\partial w}{\partial\nu}\right).$$

We can now substitute for w and $\dfrac{\partial w}{\partial\nu}$ the values given by (7·63) and (7·64): it will be seen that the components parallel to y, z give zero (to this order), and that the component parallel to x gives

$$\frac{\iota\kappa n}{2\pi}\,\frac{e^{\iota\kappa(an-\mathrm{R})}}{r}.$$

Accordingly the reflected wave is given by

$$(7\cdot67)\qquad \mathrm{X} = \frac{\iota\kappa}{2\pi r}\int n\,d\mathrm{S}\,e^{\iota\kappa(an-\mathrm{R})},$$

where R is found from (7·65) and the integral extends over the positive hemisphere. For the purpose of integration we write

$$l = \sin\theta'\cos\phi', \qquad m = \sin\theta'\sin\phi', \qquad n = \cos\theta'.$$

Then in (7·67) we have

$$an - \mathrm{R} = -r + a\,\{(1+\cos\theta)\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\}.$$

The integration of (7·67) with respect to ϕ' can be carried out at once, because

$$(7\cdot68)\qquad \int_0^{2\pi} e^{\iota\kappa a\sin\theta\sin\theta'\cos(\phi-\phi')}\,d\phi' = 2\pi \mathrm{J}_0(\kappa a\sin\theta\sin\theta').$$

Our result accordingly becomes

$$(7{\cdot}69) \qquad X = \frac{\iota\kappa a^2}{r} e^{-\iota\kappa r} \int_0^{\frac{1}{2}\pi} e^{\iota\kappa a\,(1+\cos\theta)\cos\theta'} J_0(\kappa a \sin\theta \sin\theta') \sin\theta' \cos\theta' \, d\theta'.$$

An exact evaluation of the integral would be troublesome; but it can be transformed by integration by parts. This process leads to a series of which the first term is

$$(7{\cdot}7) \qquad X = \frac{\iota\kappa a^2}{r} e^{-\iota\kappa r} \frac{J_1(\kappa a \sin\theta)}{\kappa a \sin\theta}.$$

The formula (7·7) will represent (7·69) sufficiently accurately if $(1+\cos\theta)/\sin\theta = \cot\frac{1}{2}\theta$ is regarded as small; and this is the correct assumption here, since θ is supposed to be nearly equal to π.

In the special cases $\kappa a = 9, 10$ it appears that the approximation (7·7) represents the scattered wave sufficiently well within a cone extending to about 10° from the axis.

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